Data informativity for system analysis and control part 1: analysis

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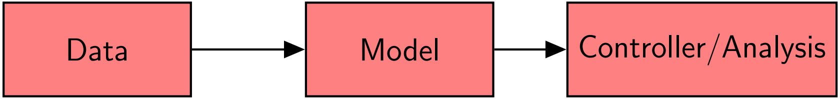
Model-based vs Data-driven.

Control and Analysis of unknown systems:



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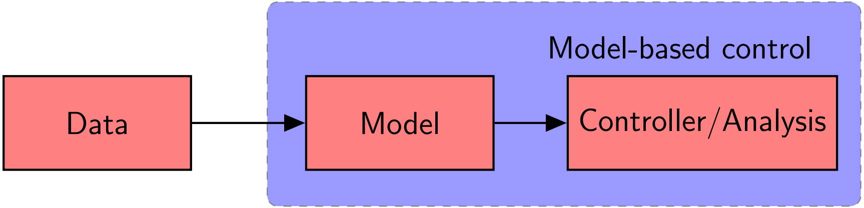

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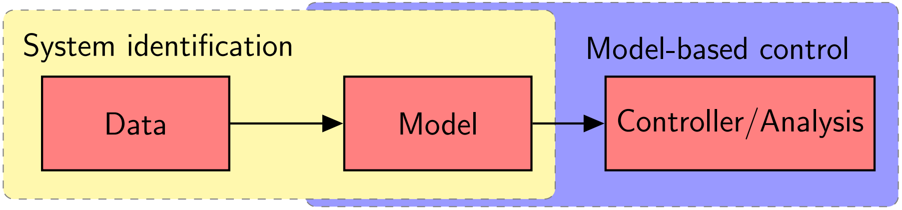

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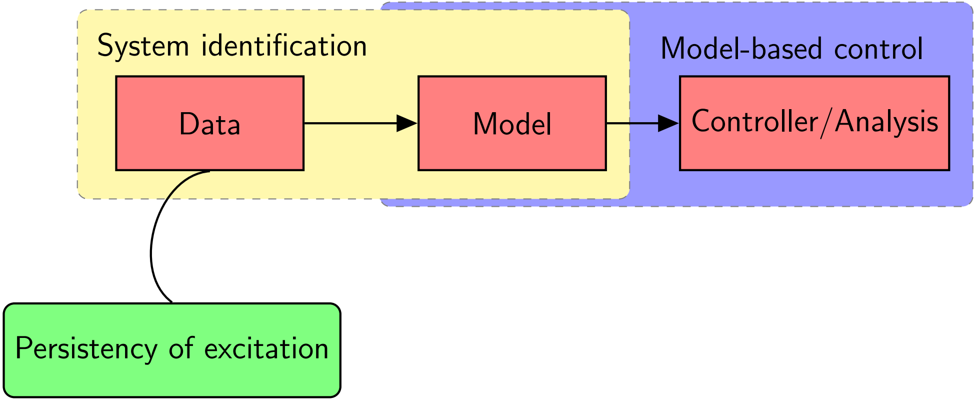

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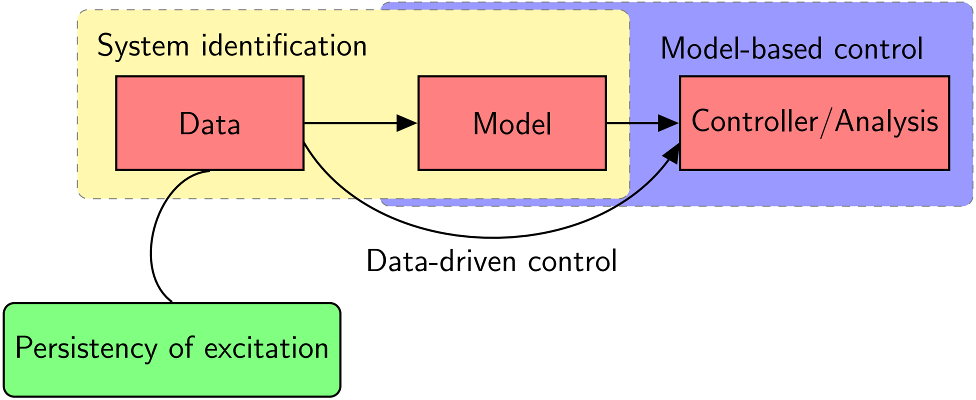

The main observation:

Consider the discrete-time, linear time-invariant system:

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Suppose that A_s and B_s are *unknown*, but that we have access to measurements on this system. We want to resolve whether the system (A_s, B_s) is stabilizable. We can only conclude that the system is stabilizable if *all* systems compatible with the measurements are stabilizable.

Contents

Not in this talk: Data-driven control, controller design, disturbances, noise.

Informativity Framework

The informativity framework for analysis.

Three important parts.

The informativity framework for analysis.

Model Class Σ

Contains the 'true' system.

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Informativity problem

Provide necessary and sufficient conditions on $\mathcal{D}$ under which the data are informative for property $\mathcal{P}$.

Informativity for stabilizability.

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\begin{aligned}
U_- &:= \begin{bmatrix} u(0) & u(1) & \cdots & u(T-1) \end{bmatrix}, & X &:= \begin{bmatrix} x(0) & x(1) & \cdots & x(T) \end{bmatrix}, \\
X_- &:= \begin{bmatrix} x(0) & x(1) & \cdots & x(T-1) \end{bmatrix}, & X_+ &:= \begin{bmatrix} x(1) & x(2) & \cdots & x(T) \end{bmatrix}.
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Then the set of all systems compatible with the data is equal to:

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Consider stabilizability, and let:

$$\Sigma_{\mathcal{P}} = \{(A, B) \mid (A, B) \text{ is stabilizable}\}$$

Informativity for stabilizability.

With data (U_-, X) as before and sets:

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Informativity for stabilizability

The data (U_-, X) are informative for stabilizability if $\Sigma_{\mathcal{D}} \subseteq \Sigma_{\mathcal{P}}$.

The informativity framework for control.

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Informativity problem for control

Provide necessary and sufficient conditions on $\mathcal{D}$ under which there exists a K such that the data are informative for property $\mathcal{P}(K)$.

Control design problem

If $\mathcal{D}$ are informative for $\mathcal{P}(K)$, find K such that $\Sigma_{\mathcal{D}} \subseteq \Sigma_{\mathcal{P}(K)}$.

Results on informativity for analysis

Data-driven Hautus test.

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$$\Sigma_{\mathrm{i/s}} = \{(A,B) \mid X_+ = AX_- + BU_-\},$$
$$\Sigma_{\text{stab}} = \{(A,B) \mid (A,B) \text{ is stabilizable}\},$$
$$\Sigma_{\text{cont}} = \{(A,B) \mid (A,B) \text{ is controllable}\}.$$

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Consider the class of discrete-time, linear time-invariant input/state systems, containing 'true' system: $x(t+1) = A_s x(t) + B_s u(t)$.

Theorem (Data-driven Hautus tests)

The data (U_-, X) are informative for controllability if and only if

$$\operatorname{rank}(X_+ - \lambda X_-) = n \quad \forall \lambda \in \mathbb{C}.$$

Similarly, the data (U_-, X) are informative for stabilizability if and only if

$$\operatorname{rank}(X_+ - \lambda X_-) = n \quad \forall \lambda \in \mathbb{C} \textit{ with } |\lambda| \geqslant 1.$$

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Let $n = 2$ and

$$x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \; x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; u_0 = 1, \; u_1 = 0.$$

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This implies that

$$X_+ = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ X_- = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ \Longrightarrow X_+ - \lambda X_- = \begin{bmatrix} 1 & -\lambda \\ 0 & 1 \end{bmatrix}$$

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Indeed:

$$\Sigma_{\mathrm{i/s}} = \left\{ \left(\begin{bmatrix} 0 & a_1 \\ 1 & a_2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \mid a_1, a_2 \in \mathbb{R} \right\}$$

Observability

Given C, consider the class of autonomous discrete-time, linear time-invariant systems, containing 'true' system: $x(t+1) = A_s x(t), \quad y(t) = Cx(t)$.

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With state measurements X as before we have sets:

$$\Sigma_{\mathsf{s}} = \{A \mid X_+ = AX_-\},$$
$$\Sigma_{\mathsf{stab}} = \{A \mid (C, A) \text{ is observable}\},$$
$$\Sigma_{\mathsf{cont}} = \{A \mid (C, A) \text{ is detectable}\}.$$

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Theorem (Data-driven Hautus tests)

The data X are informative for observability with C if and only if $\ker C \subseteq \operatorname{im} X_-$ *and for all $\lambda \in \mathbb{C}$ and $v \in \mathbb{C}^n$ we have*

$$\begin{pmatrix} X_+ - \lambda X_- \\ C X_- \end{pmatrix} v \implies X_- v = 0.$$

The data X are informative for detectability with C if and only if $\ker C \subseteq \operatorname{im} X_-$ *and for all $\lambda \in \mathbb{C}$ with $|\lambda| \geqslant 1$ and $v \in \mathbb{C}^n$ we have*

$$\begin{pmatrix} X_+ - \lambda X_- \\ C X_- \end{pmatrix} v \implies X_- v = 0.$$

Example

Let $C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$ and suppose that $X_- = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$, $X_+ = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$.

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Indeed $\ker C \subseteq \operatorname{im} X_-$ and $\begin{pmatrix} X_+ - \lambda X_- \\ C X_- \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -\lambda \\ -\lambda & 0 \\ 0 & 0 \end{bmatrix}$.

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Based on: H. J. Van Waarde, J. Eising, H. L. Trentelman, and M. K. Camlibel, “Data informativity: a new perspective on data-driven analysis and control,” *IEEE Transactions on Automatic Control*.

Extensions

> Data-driven control.

> Different model classes:

- Nonlinear systems.
- Disturbances.
- Parametrization (e.g. network structure).

> Different data:

- Noise.
- Input/output measurements

> Different properties.

Questions?