Data informativity for system analysis and control part 1: analysis

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39th Benelux Meeting on Systems and Control
Model-based vs Data-driven.

Control and Analysis of unknown systems:

Data -> Model -> Controller/Analysis
Motivation

Model-based vs Data-driven.

Control and Analysis of unknown systems:

Data → Model → Controller/Analysis

Model-based control
Model-based vs Data-driven.

Control and Analysis of unknown systems:
Motivation

Model-based vs Data-driven.

Control and Analysis of unknown systems:

Model-based control

Persistency of excitation

System identification

Data → Model → Controller/Analysis
Model-based vs Data-driven.

Control and Analysis of unknown systems:

- System identification
  - Data
  - Model
  - Controller/Analysis

- Model-based control

- Data-driven control
  - Persistency of excitation
The main observation:

Consider the discrete-time, linear time-invariant system:

\[ x(t + 1) = A_s x(t) + B_s u(t). \]
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Suppose that \( A_s \) and \( B_s \) are unknown, but that we have access to measurements on this system. We want to resolve whether the system \((A_s, B_s)\) is stabilizable.
The main observation:

Consider the discrete-time, linear time-invariant system:

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Suppose that \( A_s \) and \( B_s \) are unknown, but that we have access to measurements on this system. We want to resolve whether the system \((A_s, B_s)\) is stabilizable. We can only conclude that the system is stabilizable if all systems compatible with the measurements are stabilizable.
Contents

1. Motivation

2. Informativity Framework

3. Results on informativity for analysis

4. Conclusions

Not in this talk: Data-driven control, controller design, disturbances, noise.
Informativity Framework
The informativity framework for analysis.

Three important parts.
The informativity framework for analysis.

Model Class $\Sigma$
Contains the ‘true’ system.
The informativity framework for analysis.

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Data $\mathcal{D}$
Generated by the ‘true’ system.
Set of all systems that are compatible: $\Sigma_{\mathcal{D}}$. 
The informativity framework for analysis.

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We say that the data $\mathcal{D}$ are \textit{informative} for property $\mathcal{P}$ if $\Sigma_{\mathcal{D}} \subseteq \Sigma_{\mathcal{P}}$. 
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**Property** $\mathcal{P}$
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We say that the data $\mathcal{D}$ are *informative* for property $\mathcal{P}$ if $\Sigma_{\mathcal{D}} \subseteq \Sigma_{\mathcal{P}}$.

**Informativity problem**
Provide necessary and sufficient conditions on $\mathcal{D}$ under which the data are informative for property $\mathcal{P}$.
Informativity for stabilizability.

Consider the class of discrete-time, linear time-invariant input/state systems, containing ‘true’ system: $x(t + 1) = A_s x(t) + B_s u(t)$. 
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Consider the class of discrete-time, linear time-invariant input/state systems, containing ‘true’ system: \( x(t + 1) = A_s x(t) + B_s u(t) \).

Suppose that we measure the state and input for time 0 up to \( T \).

\[
U_\sim := [u(0) \ u(1) \ \cdots \ u(T-1)], \quad X := [x(0) \ x(1) \ \cdots \ x(T)], \\
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Then the set of all systems compatible with the data is equal to:

\[
\Sigma_D = \{(A, B) \mid X_+ = AX_\sim + BU_\sim\}.
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Consider stabilizability, and let:

\[
\Sigma_P = \{(A, B) \mid (A, B) \text{ is stabilizable}\}
\]
Informativity for stabilizability.

With data \((U_-, X)\) as before and sets:

\[
\Sigma_D = \{(A, B) \mid X_+ = AX_- + BU_-\}
\]
\[
\Sigma_P = \{(A, B) \mid (A, B) \text{ is stabilizable.}\}
\]

The data \((U_-, X)\) are informative for stabilizability if \(\Sigma_D \subseteq \Sigma_P\).
The informativity framework for control.

Let the ‘true’ system be \((A_s, B_s)\) and data \((U_-, X)\) as before.
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Suppose now that we are interested in stabilization by static state feedback: Find \(K\) such that \(A_s + B_sK\) is stable.
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Suppose now that we are interested in stabilization by static state feedback: Find \(K\) such that \(A_s + B_s K\) is stable.

We say the data \(\mathcal{D}\) is informative for property \(\mathcal{P}(K)\) if there exists a \(K\) such that \(\Sigma_{\mathcal{D}} \subseteq \Sigma_{\mathcal{P}(K)}\).
The informativity framework for control.

Let the ‘true’ system be \((A_s, B_s)\) and data \((U, X)\) as before.

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We say the data \(D\) is informative for property \(\mathcal{P}(K)\) if there exists a \(K\) such that \(\Sigma_D \subseteq \Sigma_{\mathcal{P}(K)}\).

Informativity problem for control

Provide necessary and sufficient conditions on \(D\) under which there exists a \(K\) such that the data are informative for property \(\mathcal{P}(K)\).

Control design problem

If \(D\) are informative for \(\mathcal{P}(K)\), find \(K\) such that \(\Sigma_D \subseteq \Sigma_{\mathcal{P}(K)}\).
Results on informativity for analysis
Data-driven Hautus test.

Consider the class of discrete-time, linear time-invariant input/state systems, containing ‘true’ system: $x(t + 1) = A_s x(t) + B_s u(t)$. 
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Consider the class of discrete-time, linear time-invariant input/state systems, containing ‘true’ system: 
\[ x(t + 1) = A_s x(t) + B_s u(t). \]
With data \((U_-, X)\) as before and sets:

\[ \Sigma_{i/s} = \{(A, B) \mid X_+ = AX_- + BU_-\}, \]
\[ \Sigma_{\text{stab}} = \{(A, B) \mid (A, B) \text{ is stabilizable}\}, \]
\[ \Sigma_{\text{cont}} = \{(A, B) \mid (A, B) \text{ is controllable}\}. \]
Data-driven Hautus test.

Consider the class of discrete-time, linear time-invariant input/state systems, containing ‘true’ system: \( x(t + 1) = A_s x(t) + B_s u(t) \).

**Theorem (Data-driven Hautus tests)**

The data \((U_-, X)\) are informative for controllability if and only if

\[
\text{rank}(X_+ - \lambda X_-) = n \quad \forall \lambda \in \mathbb{C}.
\]

Similarly, the data \((U_-, X)\) are informative for stabilizability if and only if

\[
\text{rank}(X_+ - \lambda X_-) = n \quad \forall \lambda \in \mathbb{C} \text{ with } |\lambda| \geq 1.
\]
Example

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The data \((U_-, X)\) are informative for controllability if and only if

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Let \(n = 2\) and

\[
x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad u_0 = 1, \quad u_1 = 0.
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The data $(U_-, X)$ are informative for controllability if and only if

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This implies that

$$X_+ = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ X_- = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \implies X_+ - \lambda X_- = \begin{bmatrix} 1 & -\lambda \\ 0 & 1 \end{bmatrix}.$$
Example

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*The data* \((U_-, X)\) *are informative for controllability if and only if*

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Indeed:

\[
\Sigma_{i/s} = \left\{ \left( \begin{bmatrix} 0 & a_1 \\ 1 & a_2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \mid a_1, a_2 \in \mathbb{R} \right\}
\]
Observability

Given $C$, consider the class of autonomous discrete-time, linear time-invariant systems, containing ‘true’ system: $x(t + 1) = A_s x(t), \quad y(t) = C x(t)$.
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Given $C$, consider the class of autonomous discrete-time, linear time-invariant systems, containing ‘true’ system: $x(t + 1) = A_s x(t)$, $y(t) = C x(t)$.

With state measurements $X$ as before we have sets:

$$\Sigma_s = \{ A \mid X_+ = AX_- \},$$
$$\Sigma_{stab} = \{ A \mid (C, A) \text{ is observable} \},$$
$$\Sigma_{cont} = \{ A \mid (C, A) \text{ is detectable} \}.$$
Observability

Given $C$, consider the class of autonomous discrete-time, linear time-invariant systems, containing 'true' system: $x(t + 1) = A_s x(t), \quad y(t) = C x(t)$.

Theorem (Data-driven Hautus tests)

The data $X$ are informative for observability with $C$ if and only if $\ker C \subseteq \im X_-$ and for all $\lambda \in \mathbb{C}$ and $v \in \mathbb{C}^n$ we have

$$\begin{pmatrix} X_+ - \lambda X_- \\ CX_- \end{pmatrix} v \implies X_- v = 0.$$

The data $X$ are informative for detectability with $C$ if and only if $\ker C \subseteq \im X_-$ and for all $\lambda \in \mathbb{C}$ with $|\lambda| \geq 1$ and $v \in \mathbb{C}^n$ we have

$$\begin{pmatrix} X_+ - \lambda X_- \\ CX_- \end{pmatrix} v \implies X_- v = 0.$$
Example

Let \( C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \) and suppose that
\[
X_- = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad
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Example

Let $C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$ and suppose that $X_- = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, $X_+ = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$.

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The data $X$ are informative for observability with $C$ if and only if $\ker C \subseteq \text{im} X_-$ and for all $\lambda \in \mathbb{C}$ and $v \in \mathbb{C}^n$ we have $\begin{pmatrix} X_+ - \lambda X_- \\ CX_- \end{pmatrix} v \implies X_- v = 0$. 
Example

Let $C = (1 \ 0 \ 0)$ and suppose that $X_- = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$, $X_+ = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$.

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The data $X$ are informative for observability with $C$ if and only if $\ker C \subseteq \text{im } X_-$ and for all $\lambda \in \mathbb{C}$ and $v \in \mathbb{C}^n$ we have $(X_+ - \lambda X_-) v \Rightarrow X_- v = 0$.

Indeed $\ker C \subseteq \text{im } X_-$ and $(X_+ - \lambda X_-) = \begin{bmatrix} 0 & 1 \\ 1 & -\lambda \\ -\lambda & 0 \\ 0 & 0 \end{bmatrix}$. 
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> Some examples illustrated these results.
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Extensions

> Data-driven control.
> Different model classes:
  • Nonlinear systems.
  • Disturbances.
  • Parametrization (e.g. network structure).
> Different data:
  • Noise.
  • Input/output measurements
> Different properties.
Questions?