Data informativity for system analysis and control part 2: design

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Introduction

Data-driven control

Goal: Obtain control laws for a system with unknown dynamics using data.

PID tuning (Ziegler and Nichols), adaptive control (Astrom *et al.*)
iterative feedback tuning (Gevers, *et al.*), unfalsified control (Safonov *et al.*).

A note on persistency of excitation

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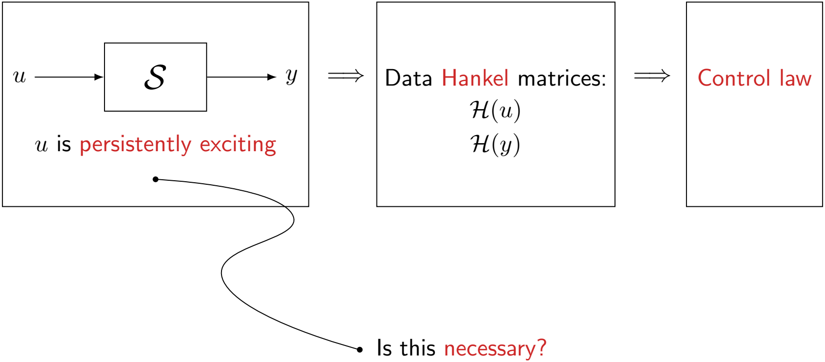
Abstract

We prove that if a component of the response signal of a controllable linear time-invariant system is persistently exciting of sufficiently high order, then the windows of the signal span the full system behavior. This is then applied to obtain conditions

Contributions by Willems, Rapisarda, Markovsky, De Moor, Coulson, Dörfler, Lygeros, Berberich, Romer, Köhler, Allgöwer, De Persis, Tesi, Rotulo, Bisoffi,...

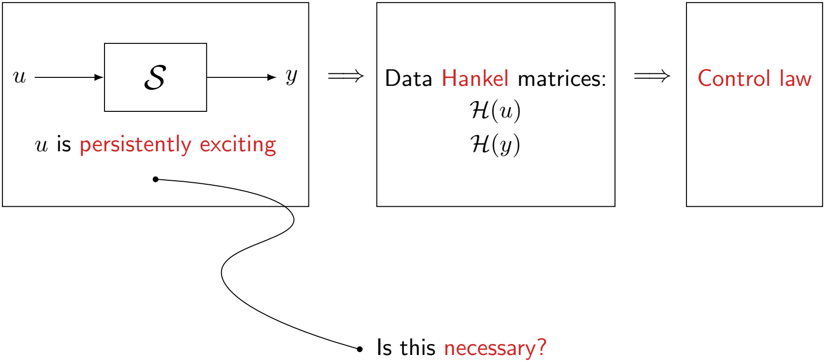
Introduction

Data-driven control



Introduction

Data-driven control



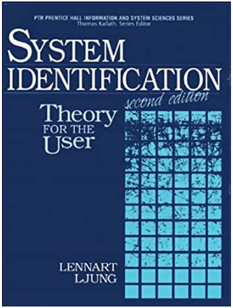
- Our goals: study data-driven control without a priori assumptions on u.
- Understand when data are informative for control.

Why do we care?

Data informativity

- Persistency of excitation guarantees that a system model can be identified.
- Data-driven control aims at identifying controllers, not models.
- Excitation is expensive.

Long term goal:
develop general theory in which concepts like data informativity are central



Outline

1. Stabilization by state feedback
2. Linear quadratic regulation
3. Extensions
   - noisy data
   - input/output data
4. Conclusions

Stabilization by state feedback

Stabilization by state feedback

A data-driven approach

Consider the linear time-invariant system

$$\boldsymbol{x}(t+1) = A_s\boldsymbol{x}(t) + B_s\boldsymbol{u}(t),$$

where $A_s \in \mathbb{R}^{n\times n}$ and $B_s \in \mathbb{R}^{n\times m}$ are unknown.

Let the following data be given:

$$\begin{aligned} X &:= \begin{bmatrix} x(0) & x(1) & \cdots & x(T) \end{bmatrix} \\ U_- &:= \begin{bmatrix} u(0) & u(1) & \cdots & u(T-1) \end{bmatrix}. \end{aligned}$$

Goal: using the data (U_-, X), find a feedback law $\boldsymbol{u} = K\boldsymbol{x}$ such that

$$\boldsymbol{x}(t+1) = (A_s + B_sK)\boldsymbol{x}(t)$$

is asymptotically stable (equivalently, $A_s + B_sK$ is Schur).

Stabilization by state feedback

A data-driven approach

Introduce the matrices:

$$X_- := \begin{bmatrix} x(0) & x(1) & \cdots & x(T-1) \end{bmatrix}, \quad X_+ := \begin{bmatrix} x(1) & x(2) & \cdots & x(T) \end{bmatrix}.$$

Note that the data and system matrices are related via: $X_+ = \begin{bmatrix} A_s & B_s \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix}$.

Define the set of all systems explaining the data:

$$\Sigma := \left\{ (A, B) \mid X_+ = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix} \right\}.$$

Definition: We say that the data (U_-, X) are informative for stabilization if there exists a feedback gain K such that $A + BK$ is (Schur) stable for all $(A, B) \in \Sigma$.

Stabilization by state feedback

An important lemma

Lemma: If the data (U_-, X) are informative for stabilization and K is a stabilizing feedback gain for all $(A, B) \in \Sigma$ then[1]

$$\operatorname{im} \begin{bmatrix} I \\ K \end{bmatrix} \subseteq \operatorname{im} \begin{bmatrix} X_- \\ U_- \end{bmatrix}.$$

Interpretation:

1. The matrix X_- necessarily has full row rank and $\operatorname{im} K \subseteq \operatorname{im} U_-$.
2. In fact, any stabilizing controller is parameterized in terms of data via

$$K = U_- X_-^\dagger,$$

where $X_-^\dagger$ is some right inverse of X_-.

Note: This parameterization was studied before[2] assuming that $\Sigma = \{(A_s, B_s)\}$.

[1] H. J. van Waarde, J. Eising, H. L. Trentelman and M. K. Camlibel, Data informativity: a new perspective on data-driven analysis and control, *IEEE Transactions on Automatic Control*, 2020.

[2] C. De Persis and P. Tesi, Formulas for data-driven control: stabilization, optimality and robustness, *IEEE Transactions on Automatic Control*, 2020.

Stabilization by state feedback

Main results

Theorem: The data (U_-, X) are informative for stabilization if and only if there exists a right inverse $X_-^\dagger$ of X_- such that $X_+X_-^\dagger$ is stable. Moreover, K is stabilizing for all $(A, B) \in \Sigma$ if and only if $K = U_-X_-^\dagger$, where $X_-^\dagger$ is as above.

Interpretation: $X_+X_-^\dagger$ is the closed-loop system matrix:

$$X_+X_-^\dagger = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix} X_-^\dagger = A + B \underbrace{U_-X_-^\dagger}_{K}.$$

Remark: A suitable right inverse is $X_-^\dagger = \Theta(X_-\Theta)^{-1}$ where $\Theta \in \mathbb{R}^{T \times n}$ satisfies

$$X_-\Theta = (X_-\Theta)^\top \quad \text{and} \quad \begin{bmatrix} X_-\Theta & X_+\Theta \\ \Theta^\top X_+^\top & X_-\Theta \end{bmatrix} > 0.$$

Stabilization by state feedback

Comparison to identification

Recall the relation:

$$X_+ = \begin{bmatrix} A_s & B_s \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix}.$$

If $\begin{bmatrix} X_- \\ U_- \end{bmatrix}$ has full row rank, we can uniquely recover $\begin{bmatrix} A_s & B_s \end{bmatrix} = X_+ \begin{bmatrix} X_- \\ U_- \end{bmatrix}^\dagger$.

Full row rank is not necessary for stabilization in general!

Example: Consider the 'true' system matrices and data

$$A_s = \begin{bmatrix} 1.5 & 0 \\ 1 & 0.5 \end{bmatrix}, \quad B_s = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad X = \begin{bmatrix} 1 & 0.5 & -0.25 \\ 0 & 1 & 1 \end{bmatrix}, \quad U_- = \begin{bmatrix} -1 & -1 \end{bmatrix}.$$

$X_+ X_-^{-1}$ is stable and $K = U_- X_-^{-1} = \begin{bmatrix} -1 & -0.5 \end{bmatrix}$ is stabilizing. However,

$$\Sigma = \left\{ \left(\begin{bmatrix} 1.5 + a_1 & 0.5a_1 \\ 1 + a_2 & 0.5 + 0.5a_2 \end{bmatrix}, \begin{bmatrix} 1 + a_1 \\ a_2 \end{bmatrix} \right) \mid a_1, a_2 \in \mathbb{R} \right\}.$$

Stabilization by state feedback

Comparison to identification

Recall the relation:

$$X_+ = \begin{bmatrix} A_s & B_s \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix}.$$

If $\begin{bmatrix} X_- \\ U_- \end{bmatrix}$ has full row rank, we can uniquely recover $\begin{bmatrix} A_s & B_s \end{bmatrix} = X_+ \begin{bmatrix} X_- \\ U_- \end{bmatrix}^\dagger$.

Full row rank is not necessary for stabilization in general!

The controller $K = U_- X_-^\dagger$ can be interpreted as a robust controller for all systems in Σ. However, the nature of uncertainty is different from classical robust control.

For identification of (A_s, B_s) we need $T \geqslant n + m$.
For stabilization we need $T \geqslant n$.

Data-driven linear quadratic regulation

Data-driven linear quadratic regulation

The problem

Consider the system

$$\boldsymbol{x}(t+1) = A_s\boldsymbol{x}(t) + B_s\boldsymbol{u}(t).$$

Given $x(0) = x_0$ and u, we consider the cost functional

$$J(x_0, u) = \sum_{t=0}^{\infty} x^\top(t)Qx(t) + u^\top(t)Ru(t),$$

where $Q = Q^\top \geqslant 0$ and $R = R^\top > 0$.

LQR problem: Determine for every x_0 an input u^* (if it exists) that minimizes the cost functional $J(x_0, u)$ under the constraint that the system is stabilized.

Fact: u^* is of the form $u^* = K^*x$, where K^* depends on the largest solution to the so-called algebraic Riccati equation (ARE).

Data-driven linear quadratic regulation

Main results

Theorem: The data (U_-, X) are informative for LQR if and only if at least one of the following two conditions hold:

1. $\Sigma = \{(A_s, B_s)\}$, and the LQR problem is solvable for (A_s, B_s, Q, R).
2. For all $(A, B) \in \Sigma$ the matrix A is stable and $QA = 0$ (thus $K^* = 0$).

Moreover, if (U_-, X) are informative then K^* is obtained as follows:

(i) The largest solution P^* to the ARE equals the unique solution to

$$\begin{aligned} &\text{maximize} \quad \operatorname{tr} P \\ &\text{subject to} \quad P = P^\top \geqslant 0 \\ &\qquad \text{and} \quad \mathcal{L}(P) := X_-^\top P X_- - X_+^\top P X_+ - X_-^\top Q X_- - U_-^\top R U_- \leqslant 0. \end{aligned}$$

(ii) There exists a right inverse $X_-^\dagger$ of X_- such that

$$\mathcal{L}(P^*) X_-^\dagger = 0.$$

Finally, the optimal feedback gain is equal to $K^* = U_- X_-^\dagger$.

Extensions

Extensions

Noisy measurements

Consider the system

$$\boldsymbol{x}(t+1) = A_s\boldsymbol{x}(t) + B_s\boldsymbol{u}(t) + \boldsymbol{d}(t),$$

where $\boldsymbol{d}(t)$ denotes unknown but bounded process noise.

Let $D_- = \begin{bmatrix} d(0) & d(1) & \cdots & d(T-1) \end{bmatrix}$ and define

$$\Sigma := \left\{ (A,B) \mid X_+ = \begin{bmatrix} A & B & I \end{bmatrix} \begin{bmatrix} X_- \\ U_- \\ D_- \end{bmatrix} \text{ for some } D_- \in \mathcal{D} \right\}.$$

Similar questions as before: e.g., when does there exist a K such that $A+BK$ is stable for all $(A,B) \in \Sigma$?

Extensions

Noisy measurements

We can write the system

$$\boldsymbol{x}(t+1) = (X_+ - D_-)X_-^\dagger \boldsymbol{x}(t)$$

as linear fractional transformation (LFT)

$$\begin{bmatrix} \boldsymbol{x}(t+1) \\ \boldsymbol{z}(t) \end{bmatrix} = \begin{bmatrix} X_+X_-^\dagger & I \\ -X_-^\dagger & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{w}(t) \end{bmatrix}$$
$$\boldsymbol{w}(t) = D_- \boldsymbol{z}(t).$$

Assumption on the noise:

$$\begin{bmatrix} D_- \\ I \end{bmatrix}^\top \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \begin{bmatrix} D_- \\ I \end{bmatrix} \geqslant 0$$

for known Q, S and R.

(U_-, X) are informative for stabilization if there exist $X_-^\dagger$ and $P > 0$ such that

$$\begin{bmatrix} * & * \\ * & * \\ * & * \\ * & * \end{bmatrix}^\top \begin{bmatrix} -P & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & Q & S \\ 0 & 0 & S^\top & R \end{bmatrix} \begin{bmatrix} I & 0 \\ X_+X_-^\dagger & I \\ 0 & I \\ -X_-^\dagger & 0 \end{bmatrix} < 0.$$

In this case, a suitable controller is $K = U_-X_-^\dagger$.

J. Berberich, A. Romer, C. Scherer, and F. Allgöwer, Robust data-driven state-feedback design, *arxiv.org/pdf/1909.04314*, 2019.

Extensions

Input and output data

Consider the system

$$\begin{aligned} \boldsymbol{x}(t+1) &= A_s\boldsymbol{x}(t) + B_s\boldsymbol{u}(t) \\ \boldsymbol{y}(t) &= C_s\boldsymbol{x}(t) + D_s\boldsymbol{u}(t). \end{aligned}$$

On the basis of input/output data, it is possible to design a dynamic controller

$$\begin{aligned} \boldsymbol{w}(t+1) &= K\boldsymbol{w}(t) + L\boldsymbol{y}(t) \\ \boldsymbol{u}(t) &= M\boldsymbol{w}(t) \end{aligned}$$

that yields a stable closed-loop system, given by

$$\begin{bmatrix} \boldsymbol{x}(t+1) \\ \boldsymbol{w}(t+1) \end{bmatrix} = \begin{bmatrix} A_s & B_sM \\ LC_s & K + LD_sM \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{w}(t) \end{bmatrix}.$$

H. J. van Waarde, J. Eising, H. L. Trentelman and M. K. Camlibel, Data informativity: a new perspective on data-driven analysis and control, *IEEE Transactions on Automatic Control*, 2020.

Conclusions

Conclusions

- Data-driven control is impossible without informative data
- We have characterized data informativity for
  - Stabilization by state feedback
  - Linear quadratic regulation
  - Stabilization by dynamic measurement feedback
- (Optimal) control design in "one shot", using finite data.
- Future work:
  - Data-driven tracking and regulation, H_∞ control
  - Experiment design (choosing u such that (U_-, X) are informative)

Data informativity: a new perspective on data-driven analysis and control

Henk J. van Waarde, Jaap Eising, Harry L. Trentelman, and M. Kanat Camlibel

***Abstract*—The use of persistently exciting data has recently been popularized in the context of data-driven analysis and control. Such data have been used to assess system theoretic**

problem are quite varied, ranging from the use of batch-form Riccati equations [9] to approaches that apply reinforcement learning [8]. Additional noteworthy data-driven control pro-

Paper to appear in IEEE Transactions on Automatic Control, available on IEEE Xplore

Thank you!